

The Simplex Method

The graphical method of solving linear programming problems is useful only for problems involving two decision variables and relatively few problem constraints.

What happens when we need more decision variables and more problem constraints?

We use an algebraic method called the simplex method, which was developed by George B. DANTZIG (1914-2005) in 1947 while on assignment with the U.S. Department of the air force.

Standard Maximization Problems in Standard Form

A linear programming problem is said to be a standard maximization problem in standard form if its mathematical model is of the following form:

Maximize the objective function

$$Z_{\max} = P = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

Subject to problem constraints of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \leq b \quad , b \geq 0$$

With non-negative constraints

$$x_1, x_2, \dots, x_n \geq 0$$

Slack Variables

“A mathematical representation of surplus resources.” In real life problems, it’s unlikely that all resources will be used completely, so there usually are unused resources.

Slack variables represent the unused resources between the left-hand side and right-hand side of each inequality.

Basic and Nonbasic Variables

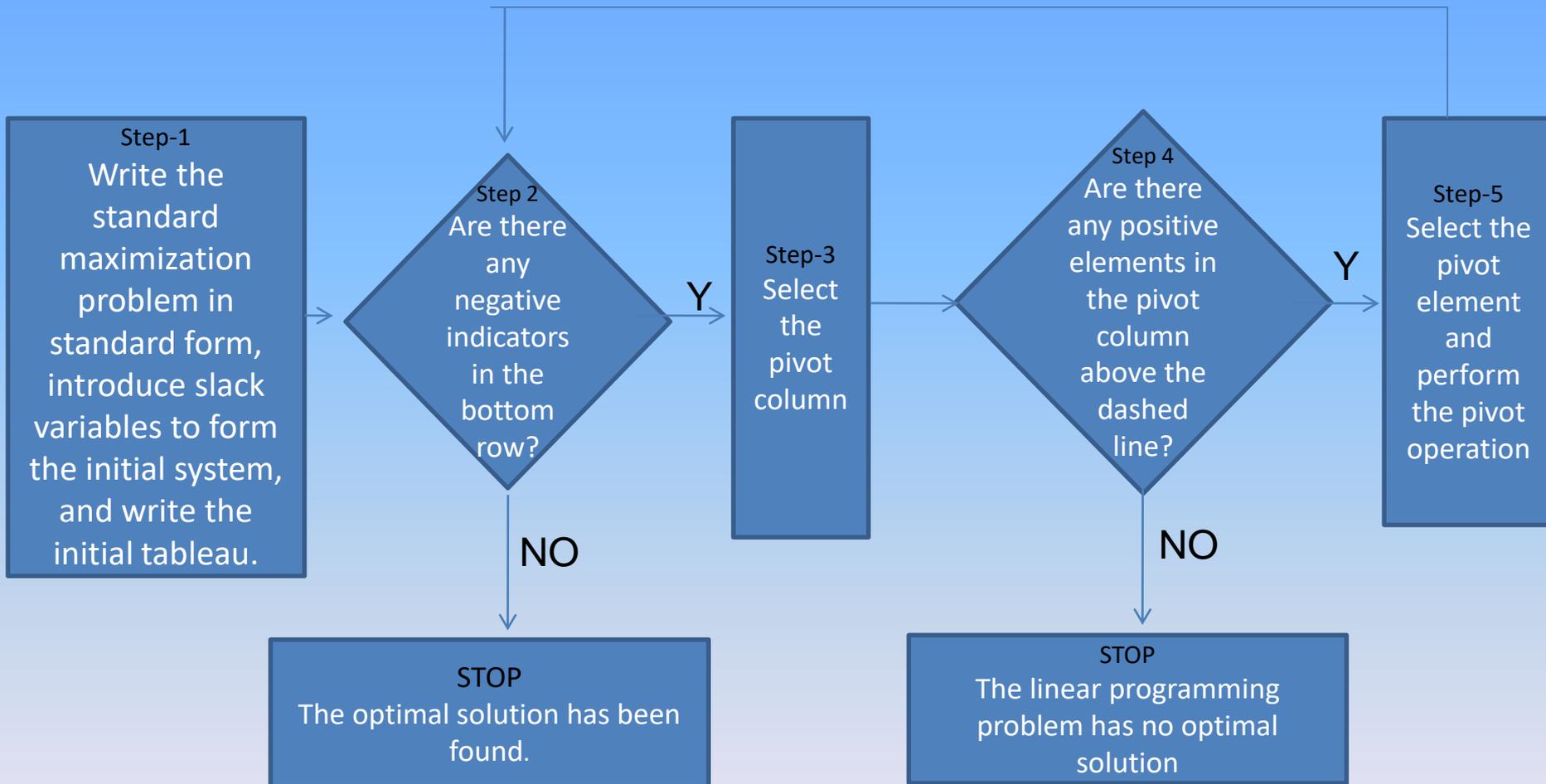
Basic variables are selected arbitrarily with the restriction that there be as many basic variables as there are equations. The remaining variables are non-basic variables.

$$x_1 + 2x_2 + s_1 = 32$$

$$3x_1 + 4x_2 + s_2 = 84$$

This system has two equations, we can select any two of the four variables as basic variables. The remaining two variables are then non-basic variables. A solution found by setting the two non-basic variables equal to 0 and solving for the two basic variables is a **basic solution**. If a basic solution has no negative values, it is a **basic feasible solution**.

SIMPLEX METHOD



Simplex algorithm for standard maximization problems

To solve a linear programming problem in standard form, use the following steps.

- 1- Convert each inequality in the set of constraints to an equation by adding **slack variables**.
- 2- Create the initial **simplex tableau**.
- 3- Select the **pivot column**. (The column with the “most negative value” element in the last row.)
- 4- Select the **pivot row**. (The row with the smallest non-negative result when the last element in the row is divided by the corresponding in the pivot column.)
- 5- Use elementary row operations calculate new values for the pivot row so that the pivot is 1 (Divide every number in the row by the **pivot number**.)
- 6- Use elementary row operations to make all numbers in the pivot column equal to 0 except for the pivot number. If all entries in the bottom row are zero or positive, this the final tableau. If not, go back to step 3.
- 7- If you obtain a final tableau, then the linear programming problem has a maximum solution, which is given by the entry in the lower-right corner of the tableau.

Pivot

Pivot Column: The column of the tableau representing the variable to be entered into the solution mix.

Pivot Row: The row of the tableau representing the variable to be replaced in the solution mix.

Pivot Number: The element in both the pivot column and the pivot row.

Simplex Tableau

Most real-world problems are too complex to solve graphically. They have too many corners to evaluate, and the algebraic solutions are lengthy. A simplex tableau is a way to systematically evaluate variable mixes in order to find the best one.

Initial Simplex Tableau

	All variables	Solution
Basic variables	coefficients	
		0

EXAMPLE

The Cannon Hill furniture Company produces tables and chairs. Each table takes four hours of labor from the carpentry department and two hours of labor from the finishing department. Each chair requires three hours of carpentry and one hour of finishing. During the current week, 240 hours of carpentry time are available and 100 hours of finishing time. Each table produced gives a profit of \$70 and each chair a profit of \$50. How many chairs and tables should be made?

STEP 1

All information about example			
Resource	Tables (x_1)	Chairs (x_2)	Constraints
Carpentry (hr)	4	3	240
Finishing (hr)	2	1	100
Unit Profit	\$70	\$50	

Objective Function

$$P = 70x_1 + 50x_2$$

Carpentry Constraint

$$4x_1 + 3x_2 \leq 240$$

Finishing Constraint

$$2x_1 + 1x_2 \leq 100$$

Non-negativity conditions

$$x_1, x_2 \geq 0$$

The first step of the simplex method requires that each inequality be converted into an equation. "less than or equal to" inequalities are converted to equations by including slack variables.

Suppose S_1 carpentry hours and S_2 finishing hours remain unused in a week. The constraints become;

$$\begin{array}{lcl} 4x_1 + 3x_2 + s_1 = 240 & & 4x_1 + 3x_2 + s_1 + 0s_2 = 240 \\ 2x_1 + x_2 + s_2 = 100 & \text{or} & 2x_1 + x_2 + 0s_1 + s_2 = 100 \end{array}$$

As unused hours result in no profit, the slack variables can be included in the objective function with zero coefficients:

$$\begin{array}{l} P = 70x_1 + 50x_2 + 0s_1 + 0s_2 \\ P - 70x_1 - 50x_2 - 0s_1 - 0s_2 = 0 \end{array}$$

The problem can now be considered as solving a system of 3 linear equations involving the 5 variables x_1, x_2, s_1, s_2, P in such a way that P has the maximum value;

$$4x_1 + 3x_2 + s_1 + 0s_2 = 240$$

$$2x_1 + x_2 + 0s_1 + s_2 = 100$$

$$P - 70x_1 - 50x_2 - 0s_1 - 0s_2 = 0$$

Now, the system of linear equations can be written in matrix form or as a 3x6 augmented matrix. The initial tableau is;

STEP 2

Basic Variables	x_1	x_2	S_1	S_2	P	Right Hand Side
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

The tableau represents the initial solution;

$$x_1 = 0, \quad x_2 = 0, \quad s_1 = 240, \quad s_2 = 100, \quad P = 0$$

The slack variables S_1 and S_2 form the initial solution mix. The initial solution assumes that all available hours are unused. i.e. The slack variables take the largest possible values.

Variables in the solution mix are called basic variables. Each basic variable has a column consisting of all 0's except for a single 1. All variables not in the solution mix take the value 0.

The simplex process, a basic variable in the solution mix is replaced by another variable previously not in the solution mix. The value of the replaced variable is set to 0.

STEP 3

Select the pivot column (determine which variable to enter into the solution mix). Choose the column with the “most negative” element in the objective function row.

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0


Pivot column

x_1 should enter into the solution mix because each unit of x_1 (a table) contributes a profit of \$70 compared with only \$50 for each unit of x_1 (a chair)

STEP 5

Select the pivot row (determine which variable to replace in the solution mix).
Divide the last element in each row by the corresponding element in the pivot column. The pivot row is the row with the smallest non-negative result.

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

Enter

Exit

Pivot number

Pivot column

Pivot row

$240/4=60$

$100/2=50$

Should be replaced by x_1 in the solution mix. 60 tables can be made with 240 unused carpentry hours but only 50 tables can be made with 100 finishing hours. Therefore we decide to make 50 tables.

Now calculate new values for the pivot row. Divide every number in the row by the pivot number.

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	4	3	1	0	0	240
x_1	1	1/2	0	1/2	0	50
P	-70	-50	0	0	1	0

$$\frac{R_2}{2}$$

Use row operations to make all numbers in the pivot column equal to 0 except for the pivot number which remains as 1.

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side	
S_1	0	1	1	-2	0	40	$-4.R_2 + R_1$
x_1	1	1/2	0	1/2	0	50	
P	0	-15	0	35	1	3500	$70.R_2 + R_3$

If 50 tables are made, then the unused carpentry hours are reduced by 200 hours (4 h/table multiplied by 50 tables); the value changes from 240 hours to 40 hours. Making 50 tables results in the profit being increased by \$3500; the value changes from \$0 to \$3500.

In this case, $x_1 = 50$, $x_2 = 0$, $s_1 = 40$, $s_2 = 0$, $P = 3500$

Now repeat the steps until there are no negative numbers in the last row.

Select the new pivot column. x_2 should enter into the solution mix.

Select the new pivot row. S_1 should be replaced by x_2 in the solution mix.

		Enter					
	Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
Exit	S_1	0	1	1	-2	0	40
	x_1	1	1/2	0	1/2	0	50
	P	0	-15	0	35	1	3500

New pivot column

New pivot row

$40/1 = 40$
 $50/0,5 = 100$

Calculate new values for the pivot row. As the pivot number is already 1, there is no need to calculate new values for the pivot row.

Use row operations to make all numbers in the pivot column equal to except for the pivot number.

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
x_2	0	1	1	-2	0	40
x_1	1	0	-1/2	3/2	0	30
P	0	0	15	5	1	4100

$$-\frac{1}{2} \cdot R_1 + R_2$$

$$15 \cdot R_1 + R_3$$

If 40 chairs are made, then the number of tables are reduced by 20 tables ($\frac{1}{2}$ table/chair multiplied by 40 chairs); the value changes from 50 tables to 30 tables. The replacement of 20 tables by 40 chairs results in the profit being increased by \$600; the value changes from \$3500 to \$4100.

As the last row contains no negative numbers, this solution gives the maximum value of P.

Result

This simplex tableau represents the optimal solution to the LP problem and is interpreted as:

$$x_1=30, \quad x_2=40, \quad s_1=0, \quad s_2=0$$

and profit or $P=\$4100$

The optimal solution (maximum profit to be made) is to company 30 tables and 40 chairs for a profit of \$4100.

Example-2

A farmer owns a 100 acre farm and plans to plant at most three crops. The seed for crops A, B, and C costs \$40, \$20, and \$30 per acre, respectively. A maximum of \$3200 can be spent on seed. Crops A, B, and C require 1, 2, and 1 workdays per acre, respectively, and there are maximum of 160 workdays available. If the farmer can make a profit of \$100 per acre on crop A, \$300 per acre on crop B, and \$200 per acre on crop C, how many acres of each crop should be planted to maximize profit?